Basic Portfolio Theory

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Key investment insights

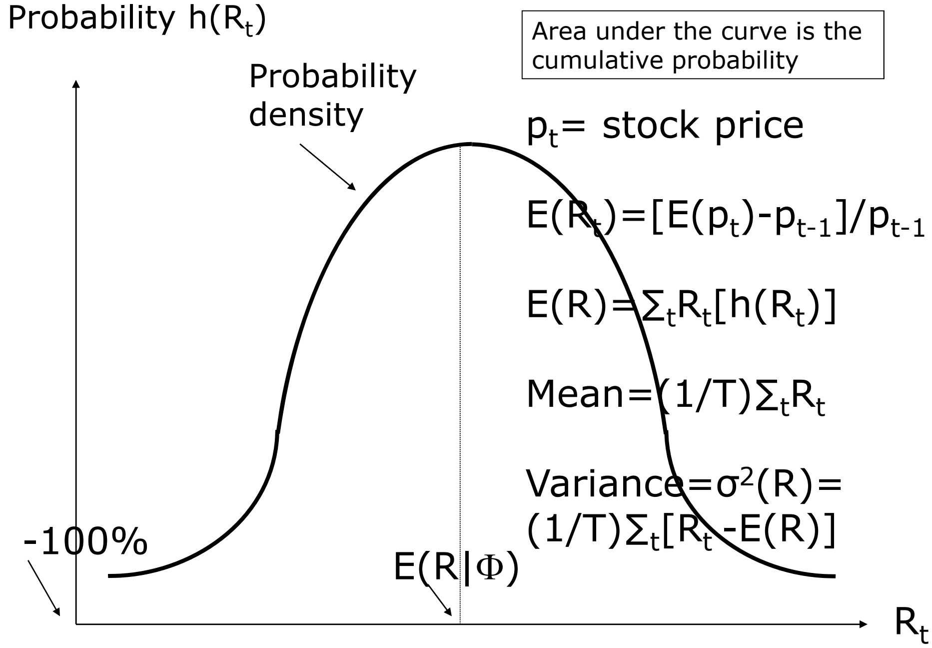
- <u>Diversification:</u>
  Always think in terms of stock portfolios rather than individual stocks
- <u>But which portfolio?</u>
  One that is highly diversified
- <u>But how much portfolio risk?</u>
  Allocate your in vestment between the risk-free asset and your diversified portfolio depending on your tolerance for risk

Optimal portfolios

- Step I: Find the "portfolio opportunity set" consisting of risky assets only
  - Cases with two risky assets
  - Arbitrary number of risky assets
  - Effect of diversification
  - Computation of optimal risky portfolio weights
  - Separation theorem
- Step II: Find the allocation between risky portfolio and risk-free assets
  - Requires specifying investor preferences

2-asset portfolio opportunities with no risk free asset

- Will show that there is a single optimal risky portfolio
- Will derive the Minimum Variance Frontier (MVF)
- The MVF is the set of portfolios with the lowest variance for a given expected return
- Will show how the shape of MVF depends on the correlation ρ between the risky securities



p_t= stock price

$E(R_t)=[E(p_t)-p_{t-1}]/p_{t-1}$

$E(R)=\Sigma_t R_t[h(R_t)]$

$\text{Mean}=(1/T)\Sigma_t R_t$

$\text{Variance}=\sigma^2(R)=(1/T)\Sigma_t[R_t-E(R)]$

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Notation

- Subscript i denotes stock i (i=1,2)

  $E_i = E(r_i)$ (expected return)

  $\sigma^2_i = \sigma^2(r_i)$ (variance)

  $\sigma_i = \sqrt{\sigma^2_i}$ (standard deviation)

  $\sigma_{ij} = \text{cov}(r_i,r_j)$ (covariance)

  $\rho_{ij} = \text{cov}(r_i,r_j)/\sigma_i\sigma_j$ (correlation coefficient)

  $-1 \leq \rho_{ij} \leq 1$

  x_i= portfolio weight of stock i

  $\Sigma_i x_i=1$ (where Σ is the summation function)

  With two stocks only: $x_2=1-x_1$

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Mean and variance of portfolio p's return:

$$E_p = x_1E_1 + x_2E_2$$

$$\sigma^2_p = x^2_1\sigma^2_1 + x^2_2\sigma^2_2 + 2x_1x_2\sigma_{12}$$

Using the definition of the correlation coeff.:

$$\sigma^2_p = x^2_1\sigma^2_1 + x^2_2\sigma^2_2 + 2x_1x_2\rho_{12}\sigma_1\sigma_2$$

Will derive the minimum variance frontier (MVF) for three different values of ρ_{12}:

$\rho_{12} = 1$ (perfect positive correlation)

$\rho_{12} = -1$ (perfect negative correlation)

$\rho_{12} = 0$ (uncorrelated assets)

Case 1: $\rho_{12} = 1$

$$\sigma^2_p = x^2_1\sigma^2_1 + x^2_2\sigma^2_2 + 2x_1x_2\sigma_1\sigma_2$$

$$\sigma^2_p = (x_1\sigma_1 + x_2\sigma_2)^2$$

$$\sigma_p = x_1\sigma_1 + x_2\sigma_2$$

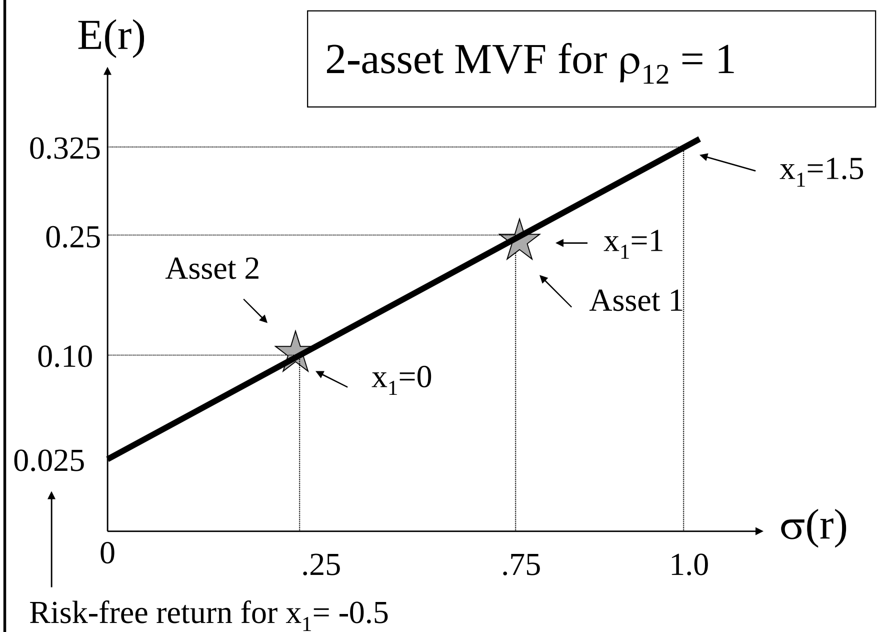
$$E_p = x_1E_1 + x_2E_2$$

Let $E_1>E_2$ and $\sigma_1>\sigma_2$ (1 most risky asset)

Since $x_2=1-x_1$, and substituting into E_p:

$$E_p = E_2 + [(E_1-E_2)/(\sigma_1-\sigma_2)](\sigma_p - \sigma_2)$$

MVF is a straight line w/positive slope



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<u>Case 2:</u> $\rho_{12} = -1$

$\sigma^2_p = x^2{}_1\sigma^2{}_1 + x^2{}_2\,\sigma^2{}_2 - 2x_1x_2\sigma_1\sigma_2$

$\sigma^2_p = (x_1\sigma_1 - x_2\,\sigma_2)^2$

Since σ_p is nonnegative, take absolute value:

$\sigma_p = |x_1\sigma_1 - x_2\,\sigma_2|$

$E_p = x_1E_1 + x_2E_2$

Note:

$\sigma_p = x_1\sigma_1 - x_2\,\sigma_2 = 0$ for $x_1 = \sigma_2/(\sigma_1 + \sigma_2)$

We just created a <u>risk free</u> asset with a <u>long</u> position in <u>both</u> risky assets:

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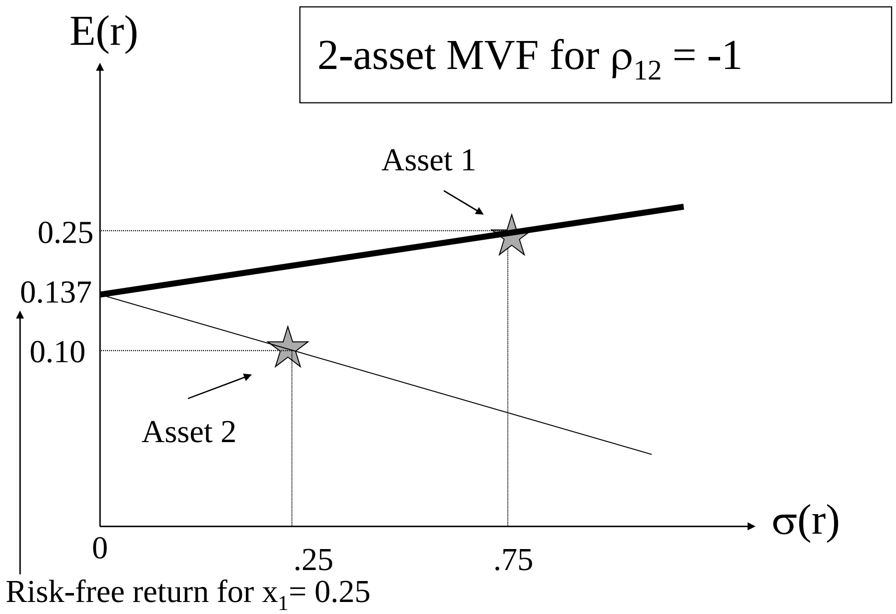


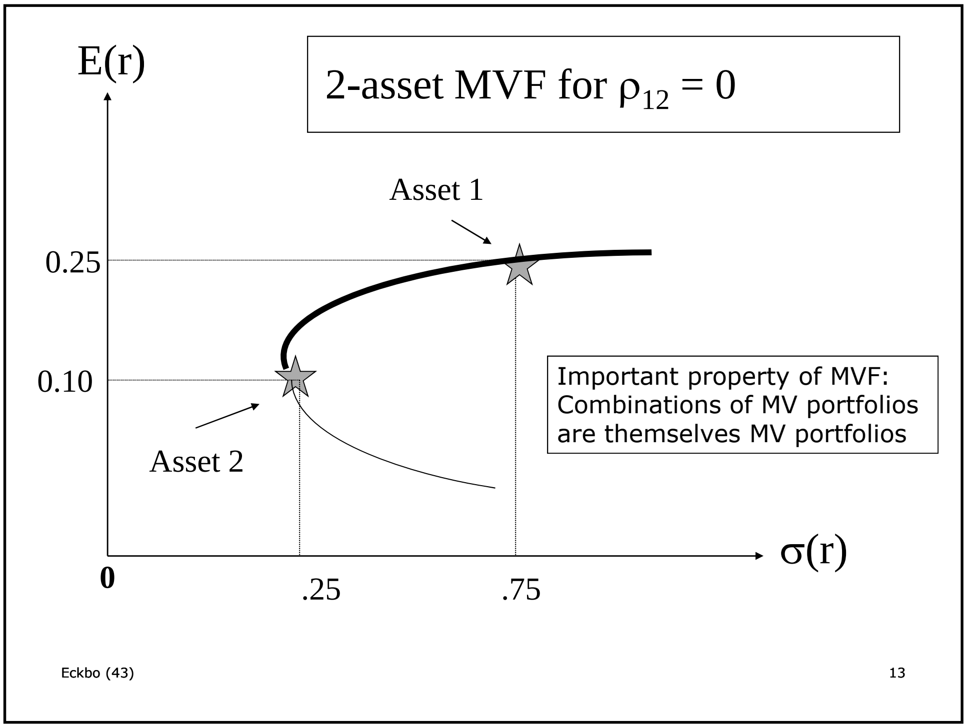
<u>Case 3</u>: $\rho_{12} = 0$

$\sigma^2_p = x^2_1\sigma^2_1 + x^2_2\,\sigma^2_2$

$\sigma_p = (x^2_1\sigma^2_1 + x^2_2\,\sigma^2_2)^{1/2}$

$E_p = x_1E_1 + x_2E_2$

MVF is no longer a straight line. It's a parabola when plotting variance and a hyperbola when plotting standard deviation. There are no risk free opportunities as long as $0 \le \rho_{12} \le 1$


2-asset MVF for $\rho_{12} = 0$
E(r)
Asset 1
0.25
0.10
Important property of MVF:
Combinations of MV portfolios
are themselves MV portfolios
Asset 2
$\sigma(r)$
0
.25
.75
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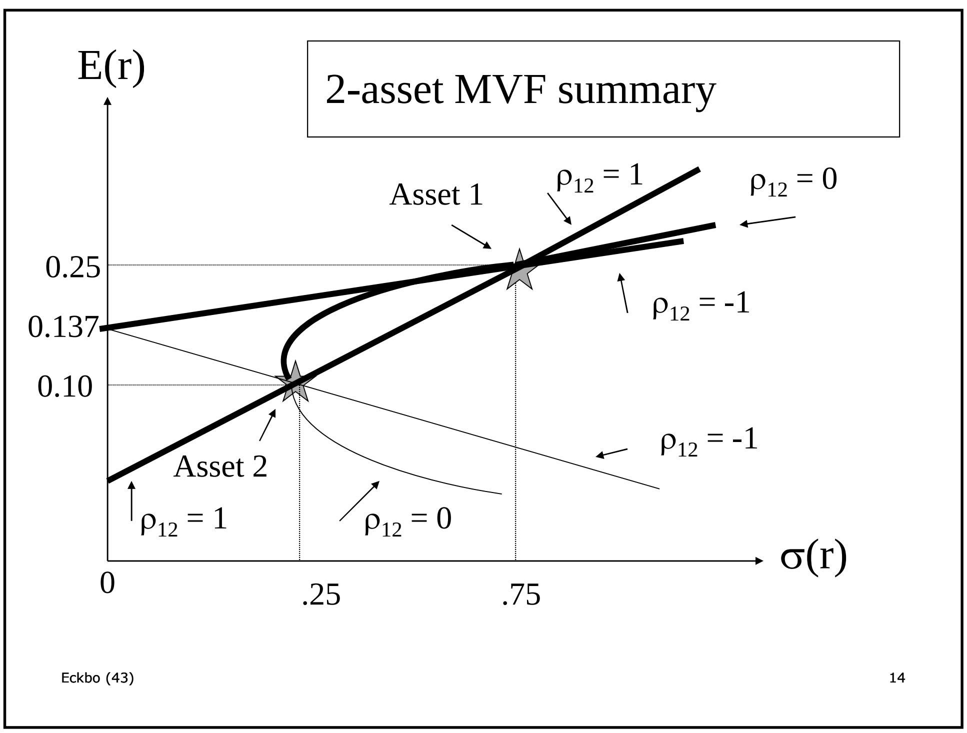
2-asset MVF summary
E(r)
Asset 1
$\rho_{12} = 1$
$\rho_{12} = 0$
0.25
$\rho_{12} = -1$
0.137
0.10
$\rho_{12} = -1$
Asset 2
$\rho_{12} = 1$
$\rho_{12} = 0$
$\sigma(r)$
0
.25
.75
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- With a risk-free asset, the weights (x) in the tangency portfolio maximizes the slope of the straight line, also called the Sharpe Ratio
- How to find these weights (x^*_1):

$\max(x)\ (E_p - r_f)/\sigma_p$ subject to

$E_p = x_1 E_1 + x_2 E_2$

$\sigma^2_p = x^2_1 \sigma^2_1 + x^2_2 \sigma^2_2 + 2 x_1 x_2 \rho_{12} \sigma_1 \sigma_2$

- Solution given two risky assets only

(e denotes excess return $r - r_f$):

$x^*_1 =$

$(E^e_1 \sigma^2_2 - E^e_2 \sigma_{12}) / [E^e_1 \sigma^2_2 + E^e_2 \sigma^2_1 - (E^e_1 + E^e_2)\ ) \sigma_{12}]$

- Example:

Asset	E_i	σ_i
1	10%	20%
2	15%	30%

Also: r_f=3% and ρ_{12}=0.5.

The Sharpe Ratio of the MVE-portfolio is

$SR_{MVE} = E^e_{MVE} / \sigma_{MVE} = 0.1250/0.2179 = 0.4359$

- Portfolio with N risky assets, i=1,..,N:
- $\Sigma_i x_i = 1$
- $E_p = \Sigma_i x_i E_i$ (sometimes we also use μ)
- $\sigma^2_p = \Sigma_i x^2_i \sigma^2_i + \Sigma_i x_i \Sigma_j x_j \sigma_{ij}$ (where $i \neq j$)

  $\Sigma_i x^2_i \sigma^2_i + \Sigma_i \Sigma_j x_i x_j \sigma_{ij}$ (where $i \neq j$)

Variance-covariance matrix **V**

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$
$\mathbf{x_1}$	σ^2_1	σ_{12}	σ_{13}
$\mathbf{x_2}$	σ_{21}	σ^2_2	σ_{23}
$\mathbf{x_3}$	σ_{31}	σ_{32}	σ^2_3

- Rule: for each σ in the matrix, premultiply by the x_i (same row) and x_j (same column) and then sum over all such products
- Thus (verify!):

  $\sigma^2_p = \Sigma_i \Sigma_j x_i x_j \sigma_{ij}$
- Note also:

  $\sigma^2_p = \Sigma_i x_i \text{cov}(r_i, \Sigma_j x_i r_j) = \Sigma_i x_i \sigma_{ip}$

  where p is the portfolio of all N assets.

  $x_i \sigma_{ip}$ is asset i's contribution to p's total risk

  σ_{ip} is therefore a marginal risk concept
- Later: $\beta_i \equiv \sigma_{ip}/\sigma^2_p$ (standardized marginal risk)

Optimal portfolio weights (“excess return over variance rule”)

- E^e = the expected excess return vector
- V = the full variance-covariance matrix
- x = optimal portfolio weights
- Step 1: Compute the raw weights: $w = E^e/V$
- Step 2: The weights w do not sum to 1. Thus, normalize: $x=w/w'I$, where I is the unit vector [1,1,1,1,1,,,1]
- Sharpe Ratio: $SR_x=x'E^e/(x'Vx)^{1/2}$

Examples of “excess return over variance” rule

- Ex 1:
  - E_A=10%, E_B=20%.
  - σ^2_A= 0.04, σ^2_B=0.09.
  - A and B are uncorrelated
  - r_F=5%
  - Compute $(E_i-r_F)/\sigma^2_i$ (i=A,B) and standardize
  - Optimal portfolio: $x_{MVE,A}$=42.86%, $x_{MVE,B}$=57.14%

- Ex 2:

Asset	E_i	σ_i
1	5%	10%
2	10%	20%
3	15%	30%

r_f=3.5%, ρ_{12}=0, ρ_{13}=0.5, ρ_{23}=0.5
$\mathbf{x}'_{MVE}$ = [0.0218 0.4619 0.5091]
$SR_{MVE}=E^e_{MVE}/\sigma_{MVE}$=0.08936/0.2163=0.4131

- Ex 3: *Add security 4*
  E_4=15% , σ_4 = 45%
  $\rho_{41}= \rho_{42}= \rho_{43}$ =0
  $\mathbf{x}'_{MVE}$ = [0.0168 0.3616 0.3924 0.2292]
  $SR_{MVE}=E^e_{MVE}/\sigma_{MVE}$=0.1302/0.1961=0.4858
- Why would anyone would hold security 4 (i.e., why is it not dominated by security 3)?

- Ex 4: *Another security 4*

E_4=5% , σ_4 = 45%

ρ_{41}= ρ_{42}= ρ_{43} = -0.2

$\mathbf{x}'_{MVE}$ = [0.1215 0.3924 0.3685 0.1175]

$SR_{MVE}=E^e_{MVE}/\sigma_{MVE}$ =0.1065/0.1646=0.4342

- Again, why would anyone would hold security 4 (this one seems even more "dominated" by security 3)?

Effect of Diversification

- What happens to σ^2_p when $N \to \infty$?
- $\sigma^2_p = \Sigma_i x^2_i \sigma^2_i + \Sigma_i \Sigma_j x_i x_j \sigma_{ij}$ (where $i \neq j$)
- Let $x_i=x_j=1/N$ (equal-weighted portfolio)
- $\sigma^2_p = (1/N^2) \Sigma_i \sigma^2_i + \Sigma_i (1/N^2) \Sigma_j \sigma_{ij}$ (where $i \neq j$)
- Substitute in the <u>average</u> σ^2_p og σ_{ij}

$AV(\sigma^2_i) = (1/N) \Sigma_i \sigma^2_i$

$AV(\sigma_{ij}) = [1/N(N-1)] \Sigma_i \sigma_{ij}$ (where $i \neq j$)

$\to \sigma^2_p = (1/N)AV(\sigma^2_i) + [(N-1)/N]AV(\sigma_{ij})$

so, as $N \to \infty$, $\sigma^2_p \to AV(\sigma_{ij})$

$N \to \infty, \quad \sigma^2_p \to AV(\sigma_{ij})$:

- In large portfolios, stocks' own-variances cancel out (is diversified away) and total portfolio risk reduces towards the average covariance
- The remaining covariance is called the portfolios systematic (nondiversifiable) risk
- We will see later that, in asset pricing models, systematic risk is the only priced risk, i.e., the only risk that generates a compensation in terms of expected return

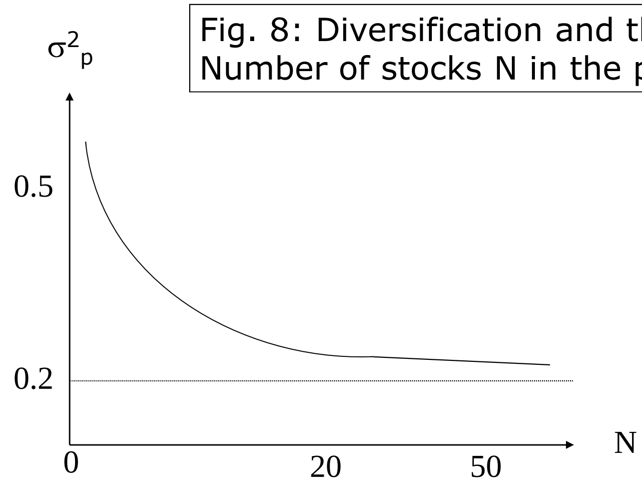


Fig. 8: Diversification and the Number of stocks N in the portfolio

II: Allocation between the risk-free asset and the optimal risky portfolio

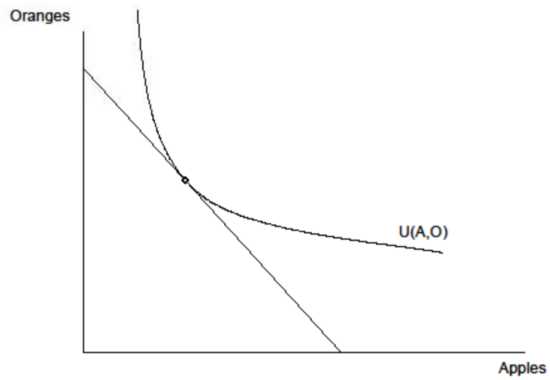
- So far, we did not introduce investor preferences (tolerance for risk)
- Now we need to model investor demand
- Will assume preferences over mean and variance of wealth W (MV-preferences)
  - Holds if returns are jointly normally distributed (only two parameters)
- Maximize expected utility: E[U(W)]

Investor's general objective:

$$\boxed{\max_{x_t} E\left[u(c_0,...,c_T)\right]}$$

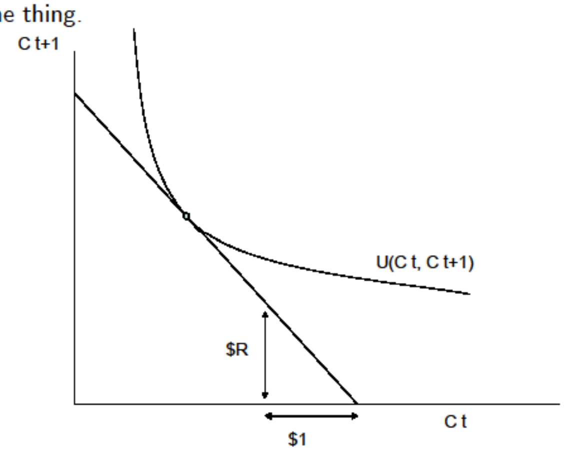
- c_t is consumption at time t
- Returns and consumption related by wealth dynamics:
  - In last period T, consume $c_T = W_{T-1}(1+r_P)$
  - Work backwards to time 0
- For simplicity, we will use:
  - 1-period time horizon
  - Mean-variance preferences over returns

maximize $U(A,O)$ s.t. $p_A A + p_O O = Y$



Marginal rate of substitution O/A = price ratio

Let's to the same thing.



Marginal rate of substitution today / tomorrow = rate of return

$$\begin{aligned} \text{utility cost of \$1 less today} &= \text{utility benefit of } R \text{ more \$ tomorrow} \\ u'(c_t) \times 1 &= E\left[\beta u'(c_{t+1}) R_{t+1}\right] \end{aligned}$$

$$1 = E\left[\beta \frac{u'(c_{t+1})}{u(c_t)} R_{t+1}\right] = E\left[m_{t+1} R_{t+1}\right]$$

A typical form:

$$\begin{aligned} u(c) &= c^{1-\gamma} \\ \gamma &= \text{coefficient of risk aversion} \end{aligned}$$

Use 1: understand interest rates.

$$R^f \approx 1 + \delta + \gamma E\left(\Delta c_{t+1}\right) - \frac{1}{2}\gamma(\gamma - 1)\sigma^2(\Delta c_{t+1})$$

When are interest rates high?

1. When people are more *impatient*, δ is high. Everyone wants to borrow, driving up rates.

2. In *good times*, $E_t\left(\Delta c_{t+1}\right)$ is high. No one wants to save, must offer them high rates. γ controls the effect – "intertemporal substitution elasticity"

3. In *safe* times. $\sigma^2(\Delta c_{t+1})$ is low. Less demand to "save for a rainy day". γ controls the effect, "risk aversion coefficient."

MV preference function over returns

$E[U(r)] = E(r) - 0.5A\,\sigma^2(r)$

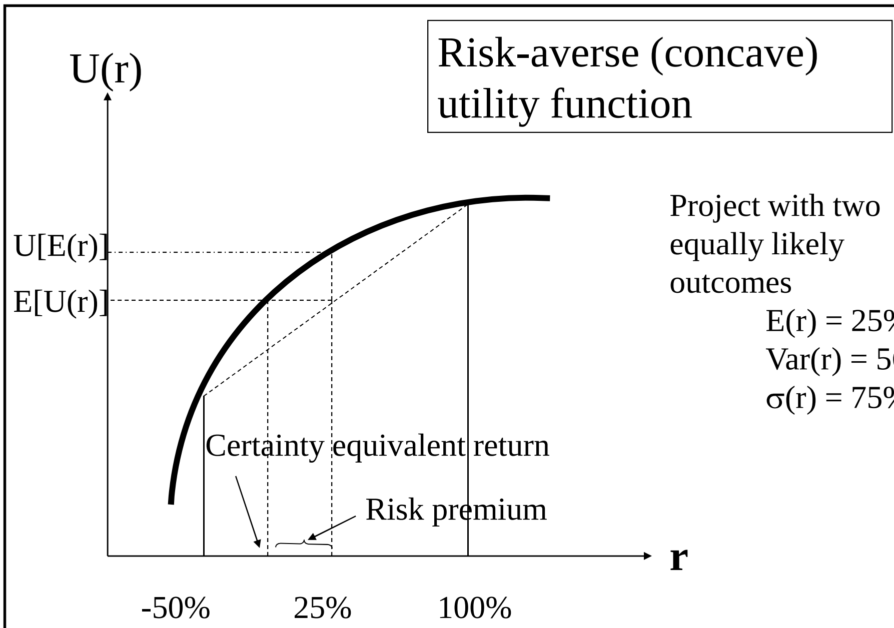
A = risk aversion coefficient: $\partial E[U]/\partial\sigma = A\sigma$

"Risk averse" investor: A>0

"Risk neutral" investor: A=0

"Risk prone" investor: A<0

The 0.5 scales the marginal utility (first derivative) and here reflects use of fractional returns, i.e., r=0.10 for 10%. If you use r=10 for 10%, then change to $0.005A\,\sigma^2(r)$.



Project with two equally likely outcomes

$E(r) = 25\%$

$Var(r) = 56\%$

$\sigma(r) = 75\%$

- Certainty equivalent return: r_{CE}=E[U(r)]
- The investor is indifferent between receiving r_{CE} with certainty or investing in the risky asset

A	0.04	0.50	0.78	1.00
r_{CE}	24%	11%	3%	-3%

- If A=0.50, will you hold a risk free asset yielding 3%?
- What A-value makes you indifferent between holding the risky and risk free assets?

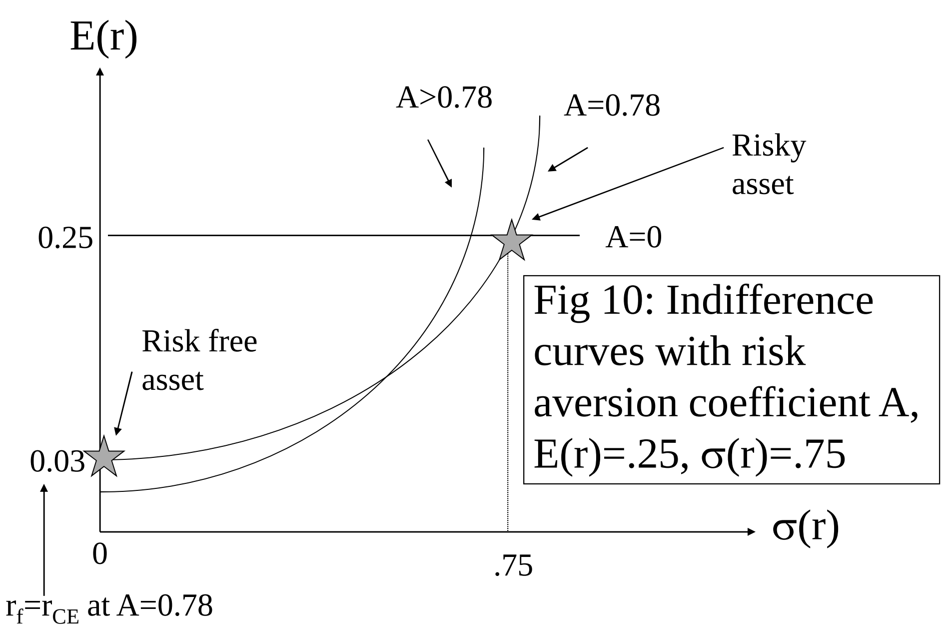


Fig 10: Indifference curves with risk aversion coefficient A, E(r)=.25, σ(r)=.75

Capital Allocation Line (CAL)

- What combinations of E and σ result from combining the risk-free and risky assets in a portfolio?
- y= portfolio weight in risky asset
- $r_p = yr+(1-y)r_f$
- $E_p=yE+(1-y)r_f = r_f + y[E-r_f]$
- $\sigma^2_p=y^2\sigma^2$ or $y= \sigma_p/\sigma$

$$\boxed{E_p= r_f +[(E-r_f)/\sigma]\sigma_p}$$

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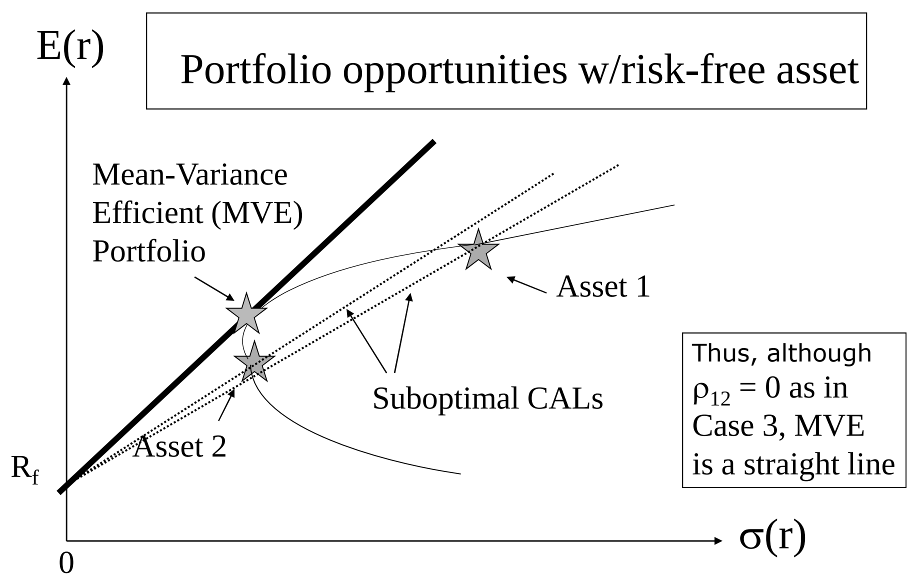

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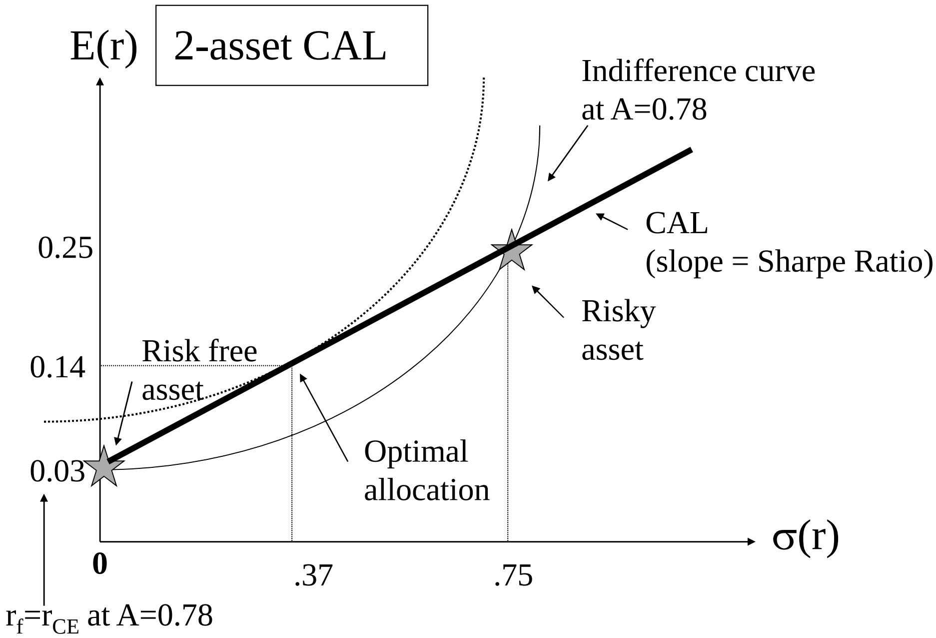

- Find the optimal portfolio weight y*:

$\max(y)\{E[U(r_p)]\}=\max(y)\{E_p-0.5A\sigma^2{}_p\}$

Substituting for E_p and $\sigma^2{}_p$ we get

$=\max(y)\{[r_f+yE(r-r_f)-0.5Ay^2\sigma^2]\}$

Solution: $y^*=E(r-r_f)/A\sigma^2$

Stock share = (1/risk aversion)(excess return/variance)

Two-fund separation theorem

- Two funds: (1) risk-free asset and (2) MVE portfolio $\mathbf{x} = \mathbf{V}^{-1}\mu^e$
- Place the fraction y of your total investment amount in the MVE portfolio
- Place the rest (1-y) in the risk-free asset
- Need to specify investor's risk aversion to determine y, while $\mathbf{x} = \mathbf{V}^{-1}\mu^e$ is independent of risk preferences (double-check)
- Thus, you can separate the computation of **x** and y. Find **x** first and then y

- In our example:

A	0.25	0.50	0.78	1.00
y^*	1.56	0.78	0.49	0.39
E_p	.37	.20	.14	.12
σ_p	1.17	.51	.37	.29

- What is the meaning of $y^*=1.56$?
- Can you ever get $y^*<0$?

Summary

- Marginal vs. total risk: The risk of an individual asset in a portfolio is its marginal (covariance) contribution to total portfolio risk
  - $\sigma^2_p = \Sigma_i \Sigma_j x_i x_j \sigma_{ij} = \Sigma_i x_i cov(r_i, \Sigma_j x_i r_j) = \Sigma_i x_i \sigma_{iM}$
- MVE portfolio: You should hold the same portfolio **x** of risky assets no matter what your risk tolerance A
  - $\mathbf{x} = \mathbf{V}^{-1}\mu^e$
- Two-fund separation: Use your risk tolerance to allocate your investment between the risky portfolio and the risk-free asset
  - $y^* = E(r - r_f)/A\sigma^2$